

Irreversible Behavior of Forced System

ROBERT H. KOHLER

Department of Physics, University of California, Los Angeles, California 90024, U.S.A.†

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Abstract

Many types of forced systems are known intuitively to undergo an irreversible behavior, in which all information about the initial conditions is eventually forgotten. These turn out to be the dynamic analogs of a system in an arbitrary initial state but connected to a static thermal reservoir and approaching equilibrium with that reservoir. The static and dynamic cases are intercombined into a single unified theory. This is based on a modified Hamiltonian and Liouville density dynamics, which explicitly includes irreversibility. Special contact transformation methods are used. The driving mechanism is generally quantal, incoherent, and of arbitrary character, not necessarily electromagnetic.

1. Introduction

The physical type of the irreversible behavior considered here is first exemplified in two well-known cases:

- (a) A system capable of thermodynamic equilibrium is connected to a static reservoir (of definite temperature, chemical potentials, etc.). The system comes to forget all of its initial conditions as it moves toward equilibrium with the reservoir.
- (b) A linear damped oscillator, electrical or mechanical, is connected to a driving source. The oscillator comes to forget all its initial conditions (as it moves for example to an a.c. steady state).

Consider also two less quantified examples, based at the moment on intuition and common experience:

- (c) A realistic system, with non-constant thermal properties, has its surface prescribed at temperature and chemical potentials that vary over the surface and with time. Then the condition of the interior, temperature, chemical potentials, etc. depend on time and position. However, they eventually lose all memory of the system's initial conditions.
- (d) A passive electrical network (with realistic non-linear temperature-dependent components) is connected to a time-varying voltage source,

† Dr. Kohler's present address is State University College, Buffalo, New York 14222, U.S.A.

within a given temperature environment (constant or prescribed in time). One intuitively expects the current response to eventually be independent of the network's initial conditions (for example its initial temperature which might be different from that of the environment). Conversely, if the network is connected to a time-varying current generator, then the voltage response will eventually be independent of initial conditions.

Broader physical examples, most of them intuitively familiar, are summarized in Section 8.

The various physical examples bear obvious similarity. The main purpose of the paper is to unify these within a single theoretical framework and provide them with a rigorous foundation.

The unification was begun in an earlier work (Kohler, 1968a, b, 1969) within the limited context of forcing by applied coherent classical electromagnetic fields. This context turns out to be inadequate to cover most physical cases. For example (c) is not covered because the driving mechanism is incoherent and not electromagnetic. Aslo (d) is not covered because the electromagnetic fields inside the network are not prescribed (Kohler, 1969, Section 14.9).

Another inadequacy of the earlier work was its basis upon idealized Hamiltonian dynamics. This has since been shown insufficient to predict irreversible phenomena (Wu, 1969).

In the present work, idealized Hamiltonian and Liouville density methods (Section 2) are modified to a realistic form that includes irreversible behavior (Section 3). Time-dependent forcing is established by decomposing a system into two Liouville densities, of which one is prescribed, and the other the response or driven (Section 5). The driving density is general and not restricted to being coherent, classical, or electromagnetic. The known irreversible behaviour of the well-insulated system (Section 4) is extended (Section 7) by special contact transformation methods (Section 6) to cover a time-dependently forced system.

2. Physical System

We begin with the abstraction of an 'infinite' system. This contains generally many different kinds of particles or fields which are ('second') quantized (Fujita, 1966).

The quantum dynamics is given by a time-dependent conservative Hamiltonian and unitary time displacement operator

$$i\hbar \partial_t U = HU, \quad U^\dagger = U^{-1}, \quad U = U(tt_0), \quad U(t_0t_0) = 1 \quad (2.1a, b, c, d)$$

A more convenient description is by the Liouville density (George *et al.*, 1972) ρ for obtaining the expectation value $\langle M \rangle$ for any operator

$$i\hbar \partial_t \rho + [\rho, H] = 0, \quad \rho(t) = U(tt_0)\rho(t_0)U^\dagger(tt_0), \quad \text{Tr } \rho = 1, \\ \langle M \rangle = \text{Tr } \rho M \quad (2.2a, b, c, d)$$

The corresponding classical formulation is presented for convenience in classical limiting cases (Finkelstein, 1973; Fujita, 1966):

$$H = H(qp), \quad \dot{q}^j = \partial H / \partial p^j, \quad \dot{p}_j = -\partial H / \partial q^j \quad (2.3a, b)$$

$$q = q(q'p't), \quad p = p(q'p't), \quad q' = q(t_0) \quad p' = p(t_0) \quad (2.4a, b)$$

$$\partial_t \rho + [\rho, H] = 0 \quad \text{or} \quad \rho(qpt) = \rho(q'(qpt_0), p'(qpt_0), t_0) \quad (2.5a, b)$$

$$\int \rho d\Gamma = 1, \quad \langle M(qpt) \rangle = \int \rho M d\Gamma \quad (2.6a, b)$$

3. Realistic System

It is assumed that the system variables are decomposed into two kinds 1, 2. These may be distinguished in any way, for example different particle types, or different space regions separated by stationary or moving boundaries. Then the Liouville density and Hamiltonian are denoted.

$$\rho = \rho_{12}, \quad H = H_{12} = H_{(1)} + H_{(2)} + H_{(12)} \quad (3.1a, b)$$

Partial densities (Jaynes, 1957a, b) may be specified as,

$$\rho_1 = \int \rho_{12} d\Gamma_2 \quad \text{or} \quad \rho_1 = \text{Tr}_2 \rho_{12} \quad (3.2a, b)$$

and by (2.6a), (2.2c) satisfy,

$$\int \rho_1 d\Gamma_1 = 1 \quad \text{or} \quad \text{Tr}_1 \rho_1 = 1 \quad (3.3a, b)$$

A special case of (3.1a) is 'uncorrelated' densities

$$\rho \equiv \rho_1 \rho_2, \quad [\rho_1, \rho_2] \equiv 0 \quad (3.4a, b)$$

For an experiment in which $\rho_2(t)$ is prescribed or maintained, then $\rho_1(t)$ is governed by its own Hamiltonian: ‡

$$H_1 = H_{(1)} + \int \rho_2 H_{(12)} d\Gamma_2 \quad \text{or} \quad H_1 = H_{(1)} + \text{Tr}_2 \rho_2 H_{(12)} \quad (3.5a, b)$$

The prescription of $\rho_2(t)$ is limited to physically realizable states for the over-all system.

Also, $\rho_2(t)$ must experimentally include at least a small random part $\rho_{2r}(\alpha t)$ in addition to the deterministic part $\rho_{2d}(t)$,

$$\rho_2(\alpha t) = \rho_{2d}(t) + \rho_{2r}(\alpha t) \quad (3.6)$$

‡ Equation (3.5) is obtained from (2.5a) or (2.1a), (3.1), (3.3), (3.4) and the classical

$$[M, N] = [M, N]_1 + [M, N]_2, \quad \int [M, N]_2 d\Gamma_2 = 0$$

The random variable(s) α are governed by a probability distribution $f(\alpha)$:

$$\int f(\alpha) d\alpha = 1 \quad \text{or} \quad \sum_i f(\alpha_i) = 1 \quad (3.7a, b)$$

Then H_1 (3.5) is the sum of deterministic H_d and small random H_r :

$$H_1(\alpha t) = H_d(t) + H_r(\alpha t), \quad \langle H_r \rangle \sim 0 \quad (3.8)$$

$$H_d = H_{(1)} + \int \rho_{2d} H_{(12)} d\Gamma_2 \quad \text{or} \quad H_d = H_{(1)} + \text{Tr}_2 \rho_{2d} H_{(12)} \quad (3.9a, b)$$

$$H_r = \int \rho_{2r} H_{(12)} d\Gamma_2 \quad \text{or} \quad H_r = \text{Tr}_2 \rho_{2r} H_{(12)} \quad (3.10a, b)$$

From $H_1(\alpha t)$ each $\rho_1(\alpha t)$ is obtained by Hamiltonian dynamics (Section 2):

$$\rho_1(\alpha t) = \rho_1(q'(qpt\alpha), p'(qpt\alpha), t_0) \quad \text{or} \quad \rho_1(\alpha t) = U(tt_0\alpha)\rho(t_0)U^\dagger(tt_0\alpha) \quad (3.11a, b)$$

The overall Liouville density $\rho_1(t)$ is then obtained averaging (Jaynes, 1957a, b)

$$\rho_1(t) = \int f(\alpha)\rho(\alpha t) d\alpha \quad \text{or} \quad \rho_1(t) = \sum_i f(\alpha_i)\rho(\alpha_i t) \quad (3.12a, b)$$

Now $\rho_1(t)$ obeys (2.6) and (2.2c, d) but *not* Hamiltonian dynamics (2.5) and (2.2a, b).

4. Well-Insulated System

The isolation of any physical system is never complete. A well-insulated system may be regarded as a subsystem ρ_1 (Section 3) under H_1 (3.5) with $H_{(12)}$ small.

Typical quasi-conserved quantities $X_{(l)}$ are:

$H_{(1)}$ total relativistic energy of 1;

$H_{(j)}$ total relativistic energy of any well-insulated subsystem of 1;

$N_{(i)}$ any particle numbers that are conserved because of restrictive walls or inadequate energy for particle creation or annihilation;

$Y_{(k)}$ other quantities, like total linear and angular momentum vectors (for a system that is free to move) (Grad, 1952a, b).

Each of the conservation laws is approximate:

$$[X_{(l)}, H_1(\alpha t)] \doteq 0, \quad X_{(l)} = H_{(1)}, H_{(j)}, N_{(i)}, Y_{(k)} \quad (4.1a, b)$$

and meaningful only within a sufficiently short time scale.‡

Any Liouville density depending only on the quasi-conserved quantities is quasi-stationary (2.5a, 2.2a)

$$\partial_t \rho_1(Xt\alpha) = \text{const. } [\rho_1(Xt\alpha), H_1(\alpha t)] \doteq 0 \quad (4.2)$$

4.1. Probability Distribution for Quasi-Conserved Quantities

P is defined through a representation choice with the quasi-conserved quantities X as generalized momenta (c) or diagonal (q):

$$d\Gamma = d\Gamma_* dX \quad \text{or} \quad \text{Tr} = \text{Tr}_* \text{Tr}_X \quad (4.1.1a, b)$$

$$P(X) = \int \rho_1 d\Gamma_* \quad \text{or} \quad P(X) = \text{Tr}_* \rho_1 \quad (4.1.2a, b)$$

These by (3.3) satisfy,

$$\int P(X) dX = 1 \quad \text{or} \quad \text{Tr}_X P(X) = 1 \quad (4.1.3a, b)$$

Also, P is quasi-stationary by (2.5a), (2.2a):

$$\begin{aligned} \partial_t P &= \partial_t \int \rho_1 d\Gamma_* = \int \partial_t \rho_1 d\Gamma_* = \int [H_1, \rho_1] d\Gamma_* \doteq \int [H_{(1)}, \rho_1] d\Gamma_* \\ &= [H_{(1)}, \int \rho_1 d\Gamma_*] = [H_{(1)}, P] = 0 \end{aligned} \quad (4.1.4)$$

$$\begin{aligned} i\hbar \partial_t P &= i\hbar \partial_t \text{Tr}_* \rho_1 = \text{Tr}_* i\hbar \partial_t \rho_1 = \text{Tr}_* [H_1, \rho_1] \doteq \text{Tr}_* [H_{(1)}, \rho_1] \\ &= [H_{(1)}, \text{Tr}_* \rho_1] = [H_{(1)}, P] = 0 \end{aligned} \quad (4.1.5)$$

4.2. Liouville Densities of Maximum Entropy

The relative likelihood for a given Liouville density is measured by its entropy (ter Haar, 1955):

$$S_1 = -k \langle \ln \rho \rangle, \quad S_1 = -k \int \rho_1 \ln \rho_1 d\Gamma_1 \quad \text{or} \quad S_1 = -k \text{Tr} \rho_1 \ln \rho_1 \quad (4.2.1a, b, c)$$

For a given $P(X)$ the maximum entropy Liouville density is:§

$$\rho_{1m} = \Omega^{-1}(X)P(X) \quad \text{or} \quad \rho_{1m} = g^{-1}(X)P(X); \quad (4.2.2a, b)$$

$\Omega(X)$ = phase volume of manifold of $d\Gamma_*$; $g(X)$ = degree of degeneracy.

This density is eventually reached ¶ by any system known to reach thermo-

‡ For a sufficiently long time-scale, all conservation laws vanish, and the insulated approximation becomes invalid.

§ The proof is a slight variation of that for 'energy shells' (ter Haar, 1955, Sections C(1), D(2)).

¶ Conversion of ρ_1 to ρ_{1m} is excluded by pure Hamiltonian dynamics (which conserves S) but is allowed by our modified Hamiltonian dynamics (ter Haar, 1955; Jaynes, 1957a, b; Wu, 1969).

dynamic equilibrium (within the same time scale as (4.1)). $P(X)$ remains equal to that of the initial density.

In the Gibbs 'inky water' model (ter Haar, 1955; Farquar, 1964) the phase fluid (taken under pure Hamiltonian dynamics) retains its volume as it thins out and fills the X 'shell' uniformly. Any small additional averaging produces true uniform filling. Such averaging is an automatic part of the (modified Hamiltonian) dynamics already discussed ‡, § (3.12).

5. Time-Dependently Forced System

Explicit time dependence in H_1 appears when the prescribed ρ_2 has such dependence.

A special case is where ρ_2 refers to classical coherent electromagnetic fields (limited by Maxwell's equations in the medium). That was the main example of an earlier paper (Kohler, 1969). The present paper covers more broad situations, where ρ_2 may be quantal, incoherent, and non-electromagnetic (Section 8.3). Also the surfaces of 1 need not be fixed or rigid, as (3.4), (3.5) remain meaningful and valid without such restriction. (Sections 8.2, 8.4).

6. Contact Transformation to a Prescribed Hamiltonian

In idealized Hamiltonian dynamics (Section 2) with $H_r \equiv 0$, a contact transformation is specified by a generating function (Finkelstein, 1973) or a unitary operator. The result is a transformed representation with transformed Hamiltonian \tilde{H} .

It has been shown quantum dynamically possible to reverse the procedure or first choose the transformed representation and \tilde{H} , and later find a suitable unitary G (Kohler, 1969):

$$G(t) = U(tt_0)U_a\tilde{U}^\dagger(tt_0), \quad \partial_t U_a \equiv 0 \quad (6.1)$$

$$\tilde{U} = \text{time displacement operator for } \tilde{H}. \quad (6.2)$$

Equivalently, (6.1, 6.2) is a general decomposition of any unitary (contact) transformation.

‡ The more conventional methods of averaging (ter Haar, 1955; Farquar, 1964) are (roughly) included as special choices:

- (a) 'ergodic' (time averaging) $H_r(\alpha) = \lambda(\alpha)H_a$, λ small;
- (b) classical 'coarse-graining' $H_r(\alpha\beta) = \sum_i (\lambda_i(\alpha_i)q_i + k_i(\beta_i)p_i)$;
- (c) quantum 'coarse-graining' $H_r(\alpha_{ij}) = \sum_{ij} (|i^*\rangle\langle I^*| + |I^*\rangle\langle i^*|)\lambda_{ij}(\alpha_{ij})$, where * denotes variables of Tr_* (4.1.1).

§ The averaging, although due to random part of the interactions $H_{(12)}$ does not influence the gross relaxation rate of ρ_1 within the limited time scale considered (Blatt, 1959). Analysis in which the relaxation rate is due to the interaction with a reservoir, refer to a stronger $H_{(12)}$ or longer time-scale, and are not appropriate to the 'insulated' approximation (Mayer, 1961; Bergman & Lebowitz, 1955; Lebowitz & Frisch, 1957; Gross & Lebowitz, 1956; Klein, 1952).

It is now shown that the corresponding reversed procedure is available classically. An arbitrary contact transformation is:

$$\begin{array}{lll}
 H(qpt), & q = q(\tilde{q}\tilde{p}t) \quad p = p(\tilde{q}\tilde{p}t), & \tilde{H}(\tilde{q}\tilde{p}t) \\
 \text{original} & \text{Poisson-Bracket Invariant} & \text{transformed} \\
 \text{Hamiltonian} & \text{transformation equations} & \text{Hamiltonian}
 \end{array} \quad (6.3)$$

This may always be decomposed into three successive transformations:

$$\begin{array}{lllllll}
 H(qpt) & q = q(q'p't) & H' \equiv 0 & q' = q'(\tilde{q}'\tilde{p}') & \tilde{H}' \equiv 0 & \tilde{q}' = \tilde{q}'(\tilde{q}\tilde{p}t) & \tilde{H}(\tilde{q}\tilde{p}t) \\
 & p = p(q'p't) & & p' = p'(\tilde{q}'\tilde{p}') & & \tilde{p}' = \tilde{p}'(\tilde{q}\tilde{p}t) & \\
 \text{(a)} & \text{(b)} & \text{(c)} & \text{(d)} & \text{(e)} & \text{(f)} & \text{(g)}
 \end{array} \quad (6.4)$$

(a) Original Hamiltonian; (b) Equations of motion under \tilde{H} ; (c) Hamiltonian for constants; (d) Transformation to initial conditions under \tilde{H} ; (e) Hamiltonian for constants; (f) Inverse equations of motion under \tilde{H} ; (g) Transformed Hamiltonian

as H, \tilde{H} by (2.4) uniquely determine (b), (f), which uniquely determine (d). Now if H, \tilde{H} and (d) are independently chosen, the transformation (6.4) or (6.3) is completely determined. This is the desired classical result.

A related classical proof based on the known equation involving the generator f ,

$$\Sigma p^j \dot{q}^j - H = \Sigma \tilde{p}'^j \dot{\tilde{q}}'^j - \tilde{H} + df_{q\tilde{q}}/dt \quad (6.5)$$

Solution for f yields (Prange, 1935):

$$f_{q\tilde{q}} = \int_{t_0}^t (\Sigma p^j \dot{q}^j - H) dt + g(q'\tilde{q}') - \int_{t_0}^t (\Sigma \tilde{p}'^j \dot{\tilde{q}}'^j - \tilde{H}) dt \quad (6.6)$$

Action integral or Hamiltonian Jacobi principle function	Integration constant	Action integral or Hamiltonian Jacobi principle function
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$$f_{q\tilde{q}}(q\tilde{q}t) = f_{qq}(qq't) + f_{q\tilde{q}'}(q'\tilde{q}') + f_{\tilde{q}\tilde{q}'}(\tilde{q}'\tilde{q}t) \quad (6.7)$$

Thus the generator $f_{q\tilde{q}}$ is uniquely determined by H, \tilde{H} and $f_{q\tilde{q}'}$. (The method of adding generators for composite transformations is discussed by Schwinger, 1970.)

7. Irreversible Property of Forced System

For a well-insulated system that can reach thermodynamic equilibrium, a consequence of Section 4 is the convergence: If

$$P_{1\text{I}}(Xt_0) - P_{1\text{III}}(Xt_0) \doteq 0 \quad \text{then} \quad \rho_{1\text{I}}(t) - \rho_{1\text{III}}(t) \rightarrow \doteq 0 \quad (7.1a, b)$$

To specify a suitable forced system it is convenient first to regard the well-insulated system I as factored into two partial densities with a single $\rho_4(t)$ prescribed for both $\rho_{1I}(t)$ and $\rho_{1III}(t)$:

$$\rho_1 = \rho_3 \rho_4, \quad H_3(\beta t) = H_{3d}(t) + H_{3r}(\beta t) \quad (7.2a, b)$$

These hold only with suitable limitations on what degrees of freedom can be prescribed (during the natural approach to equilibrium) and how far apart the initial $\rho_{3I}(t_0)$, $\rho_{3III}(t_0)$ can be. Thereafter the analysis of Section 3 applies, with suitable notation change.

The forced system is then specified by experimentally prescribing ρ_4 in a more arbitrary way $\bar{\rho}_4(t)$:

$$\bar{\rho}_1 = \bar{\rho}_3 \bar{\rho}_4, \quad \bar{H}_3(\gamma t) = \bar{H}_{3d}(t) + \bar{H}_{3r}(\gamma t) \quad (7.3a, b)$$

In order to compare the behaviors under \bar{H}_3 vs. H_3 we note the existence of contact transformation that connects \bar{H}_{3d} and H_{3d} but leaves initial conditions unchanged. Take (6.4d) or U_a of (6.1) as the identity:

$$\bar{H}_{3d}(\bar{q}\bar{p}t), \quad \bar{q} = \bar{q}(qpt), \quad \bar{p} = \bar{p}(qpt), \quad H_{3d}(qpt) \quad (7.4a, b, c)$$

$$\bar{q}^j(t_0) = q^j(t_0), \quad \bar{p}^j(t_0) = p^j(t_0) \quad (7.5)$$

$$\bar{H}_{3d}(\bar{q}\bar{p}t), \quad G(t), \quad G(t_0) = 1, \quad H_{3d}(t) \quad (7.6)$$

The influence on the total Hamiltonians is:

$$\begin{aligned} & \bar{H}_{3d}(\bar{q}\bar{p}t) \\ & + \bar{H}_{3r}(\bar{q}\bar{p}\gamma t) \end{aligned} \quad \bar{q} = \bar{q}(qpt) \quad \bar{p} = \bar{p}(qpt) \quad \begin{aligned} & H_{3d}(qpt) \\ & H_{3r}(q\bar{p}\gamma t) \end{aligned} \quad (7.7a, b, c)$$

$$\bar{H}_{3d}(t) + \bar{H}_{3r}(\gamma t), \quad G(t), \quad H_{3d}(t) + H_{3r}(\gamma t) \quad (7.8a, b, c)$$

where H_{3r} denote small random terms (transformed from \bar{H}_{3r}).

To compare asymptotic behaviors under H_3 and \bar{H}_3 for the same two starting densities,

$$\rho_{3I}(t_0) = \bar{\rho}_{3I}(t_0), \quad \rho_{3III}(t_0) = \bar{\rho}_{3III}(t_0) \quad (7.9a, b)$$

it is convenient to define

$$\Delta\rho_3(t) = \rho_{3I}(t) - \rho_{3III}(t), \quad \Delta\bar{\rho}_3(t) = \bar{\rho}_{3I}(t) - \bar{\rho}_{3III}(t) \quad (7.10a, b)$$

which by (7.5), (7.6d) satisfy

$$\Delta\rho_3(t_0) = \Delta\bar{\rho}_3(t_0) \quad (7.11)$$

By (7.1), (3.2, 3.12), it follows that,

$$\Delta\rho_3(t) = \int f_\beta(\beta) \Delta\rho_3(\beta t) d\beta \rightarrow \doteq 0 \quad (7.12)$$

independent of f_{ρ_2} and $H_r(\beta t)$ except that H_r be random and small. Compute now $\Delta\bar{\rho}_3$ classically with (7.7),

$$\begin{aligned}\Delta\bar{\rho}_3(t) &= \int f_\gamma(\gamma) \Delta\bar{\rho}_3(\bar{q}\bar{p}\gamma t) d\gamma \\ &= \int f_\gamma(\gamma) \Delta\bar{\rho}_3(\bar{q}(qpt), \bar{p}(qpt), \gamma, t) d\gamma \\ &= \int f_\gamma(\gamma) \Delta\rho_3(qp\gamma t) d\gamma = \Delta\rho_3(t)\end{aligned}\quad (7.13)$$

and quantum dynamically with (7.8),

$$\begin{aligned}\Delta\bar{\rho}_3(t) &= \int f_\gamma(\gamma) \Delta\bar{\rho}_3(\gamma t) d\gamma = G \int f_\gamma(\gamma) G^\dagger \Delta\bar{\rho}_3(\gamma t) G d\gamma G^\dagger \\ &= G \int f_\gamma(\gamma) \Delta\rho_3(\gamma t) d\gamma G^\dagger = G\Delta\rho_3(t)G^\dagger\end{aligned}\quad (7.14)$$

Thus from (7.12), (7.13) and (7.14) we obtain the desired result,

$$\Delta\bar{\rho}_3(t) \rightarrow \doteq 0 \quad (7.15)$$

The summarized results for

$$P_{\text{I}}(Xt_0) - P_{\text{III}}(Xt_0) \doteq 0 \quad (7.16)$$

are

$$\text{(unforced)} \quad \rho_{\text{I I}}(t) - \rho_{\text{I III}}(t) \rightarrow \doteq 0, \quad \rho_{\text{3 I}}(t) - \rho_{\text{3 III}}(t) \rightarrow \doteq 0 \quad (7.17a, b)$$

$$\text{(forced)} \quad \bar{\rho}_{\text{I I}}(t) - \bar{\rho}_{\text{I III}}(t) \rightarrow \doteq 0, \quad \bar{\rho}_{\text{3 I}}(t) - \bar{\rho}_{\text{3 III}}(t) \rightarrow \doteq 0 \quad (7.18a, b)$$

where (7.18) has been inferred indirectly from the known (7.17).

8. Results and Applications

8.1. Summary of the General Theory

A system $\rho_1(t)$ capable of thermodynamic is assumed factorable $\rho_1(t) = \rho_3(t)\rho_4(t)$ with $\rho_4(t)$ independent of two different (sufficiently close) $\rho_3(t)$.

Now if $\rho_4(t) = \bar{\rho}_4(t)$ in a forcing experiment is prescribed in a more arbitrary way, then the same two ρ_3 starting densities will still converge. Equivalently, the eventual $\rho_4(t)$ depends only on the overall I systems initial value of quasi-conserved quantities like total relativistic energy, energies of insulated subsystems, any conserved particle numbers, linear and angular momenta, etc. The relevant conserved quantities depend on the particular system and time-scale.

8.2. Movable and Stationary Systems

The general rules given refer to moving as well as stationary systems. For moving systems, a statistical mechanics and thermodynamic has been developed (Grad, 1952a, b).

8.3. Forcing Over the Full Volume

If the degrees 3 and 4 occupy the same region of space, then the range of 3 is restricted for 4 to be prescribed independently.

An example is where 4 refers to a classical coherent electromagnetic field, and 3 the medium. The medium's temperature response changes the resistivities and susceptibilities. Field prescribability is meaningful in a suitably restricted sense (Kohler, 1969, Sections 14, 15) but, if achieved, the medium's response eventually becomes independent of initial conditions.

In the present work the electromagnetic field $\rho_4(t)$ may be quantal and incoherent. Medium $\rho_3(t)$ responses on an atomic level (changes in occupation numbers, etc.) may be considered.

Also, in the present work, the driving agency $\rho_4(t)$ is not necessarily electromagnetic. It may for example be the medium, while the electromagnetic field is the response.

One case is a strong acoustic wave. The electromagnetic field response (coupled thermally or piezo elastically) will eventually be independent of initial conditions (except the overall system's total energy, etc. (Section 8.1).

8.4. Forcing by a Generalized Reservoir

We suppose 3 to be small system with a geometrically separate large system 4 called a 'generalized reservoir'. For the composite 1 (3 + 4) we find‡ at the initial time

$$P_1(X) = \int P_3(X - \hat{X})P_4(\hat{X})d\hat{X} \quad \text{or} \quad P_1(X') = \Sigma_{\hat{X}'} P_3(X' - \hat{X}')P_4(\hat{X}') \quad (8.4.1)$$

Under the assumptions that the width of P_3 is much less than the width of P_4 (the experimental error is proportional to the quantity measured), $P_1(Xt_0)$ is approximately independent of P_3 and hence of ρ_3 . Thus ρ_3 eventually becomes independent of its initial conditions.

The boundary between 3 and 4 (a physical or a convenient mathematical surface) is not necessarily fixed or rigid. Cases where the generalized reservoir 4 moves, compresses, or distorts 3 are included.

8.5. The Generalized Reservoir is Also Electromagnetic

The prescribed Liouville density $\rho_4(t)$ necessarily includes electromagnetic degrees of freedom (among others). The reservoir 4 thus imposes an electromagnetic field density (at least only thermal) onto 3, eventually independent of 3's initial conditions.

8.6. Forcing at the Surface

For some special cases, the interaction between (small) system 3 and reservoir 4 takes place strictly at the surface of the two. Thus the forcing of

‡ Equation (8.4.1) is obtained from (4.1.2) by $X_1 \doteq X_3 + X_4$ and the variable change, $X_4 = \hat{X}$, $X_3 = X_1 - \hat{X}$.

$\rho_3(t)$ is given by its (time- and space-dependent) surface conditions. These impose upon $\rho_3(t)$ a dynamic state that is eventually independent of all of its initial conditions (except, energy, etc.) as a special case of Section 8.4. An example is Section 1(c).

8.7. *The Approximation of Local Equilibrium*

Local equilibrium is a special case with well-defined space time dependent entropy per unit volume and entropic intensive parameters.

The various local parameters are: densities of entropy, energy, particle numbers; hydrodynamic velocities; pressure, stress, strain; entropic or energy intensive parameters; affinities, fluxes, etc. These parameters are interrelated by equilibrium and irreversible thermodynamics (Callen, 1960; de Groot & Mazur, 1962; Meixner & Reik, 1959), and may be reduced to a small set of independent parameters.

If the subsystems 1, 2, 3 are each in local equilibrium, then their Liouville densities may be given in standard forms depending only on the minimal set. A simplified example is a medium without the hydrodynamic motion, with a local grand canonical Liouville density near r_l, t specified completely by F^l and volume size ΔV^l :

$$\rho_l = \exp \left\{ -k^{-1} \langle S^l \rangle - \sum_i F_i^l \langle X_i^l \rangle - k^{-1} \sum_i F_i^l X_i^l \right\}$$

with

$$F_i = \partial \langle S \rangle / \partial \langle X_i \rangle = \text{entropic intensive parameter.}$$

$$\langle S^l \rangle - \sum F_i^l \langle X_i^l \rangle = \Delta v^l f(F^l)$$

$$\rho = \prod_l \rho_l \quad (\text{over the full volume})$$

Furthermore, surface forcing of 3, if appropriate, is specified by the local parameters of 4 in a thin layer adjacent to its surface. Then 3's local parameters eventually become independent of its initial conditions. An example is in Section 1(c).

8.8. *Related Studies from Non-Equilibrium Thermodynamics*

There have been various efforts at obtaining the relations of non-equilibrium thermodynamics from an extremum principle, primarily of the entropy. To the extent that these are successful, they also give implication of a unique dynamic state for specified boundary conditions.

The direct entropy extremum principle has been fruitful for time-independent boundary conditions or steady state with the Onsager kinetic coefficients L_{ij} taken as constants (Prigogine, 1955; de Groot & Mazur, 1962; Donnelly *et al.*, 1965). For non-constant coefficients and special cases (isotropy and no hydrodynamic motion) a recent theory yields the steady state for extremum of another quantity—not the entropy (Van Kampen, 1973).

One proof indicates the non-existence of a general extremum principle for steady states (Donnelly *et al.*, 1965, Part 6, p. 283).

A modified 'local potential' or double variational method has been fruitful for non-constant coefficients and time-dependent or independent boundary conditions (Donnelly *et al.*, 1965).

In a different sense, the present paper is also based on an entropy extremum principle. This is applied directly to a system in equilibrium (Section 4.2) and transferred to a forced system by contact transformation methods (Section 7).

8.9. Quasi-Stationary Circuit

The physical situation is discussed in Section 1(d).

The response of a quasi-stationary circuit may be treated with the help of Fig. 1. The electromagnetic fields in A are prescribable because A and the major part of the load N are controlled in temperature (and other intensive parameters). The prescribed Liouville densities $\rho_4(t)$ are the total density of

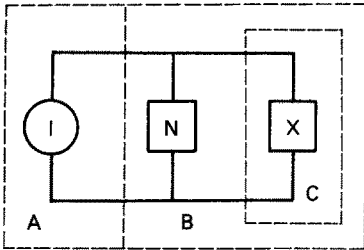


Figure 1a.—The quasi-stationary circuit under a time-dependent voltage. I . Current generator with arbitrary time variation. Effective resistance of $N \ll$ effective resistance of X .

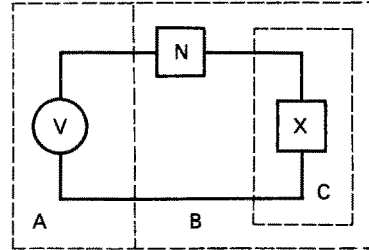


Figure 1b.—The quasi-stationary circuit under a time-dependent current. V . Voltage generator with arbitrary time variation. Effective resistance of $N \gg$ effective resistance of X .

X External load.

N Internal load.

A Region of controlled electromagnetic field.

AB Region of controlled temperature and other intensive parameters.

C Region of driven circuit where temperature and other intensive parameters are not controlled.

A and the total excepting electromagnetic density of B . The variable density $\rho_3(t)$ is the electromagnetic density of B and the total density of C . The latter includes the network. Its response by Section 8.1 is eventually independent of its initial conditions.

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